Chase Joyner

901 Homework 3

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Problem 5.6

Suppose $X \in L_1$ and A and A_n are events.

(a) Show $\int_{\{|X|>n\}} XdP \to 0$.

Solution: Given $X \in L_1$, $E[X^+]$ and $E[X^-]$ are both finite and hence $E[|X|]$ is finite, i.e. $|X| \in L_1$. Note that $XI_{\{|X|>n\}} \to 0$ as $n \to \infty$ and $|XI_{\{|X|>n\}}| \leq |X|$ for all n. Then, since $|X| \in L_1$, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\{|X|>n\}} XdP = \int_\Omega \lim_{n\to\infty} XI_{\{|X|>n\}} dP = \int_\Omega 0dP = 0.$$

(b) Show that if $P(A_n) \to 0$, then $\int_{A_n} XdP \to 0$.

Solution: First let X be a nonnegative random variable and consider for some M,

$$\begin{aligned}\int_{A_n} XdP &= \int_{A_n \cap \{X \leq M\}} XdP + \int_{A_n \cap \{X>M\}} XdP \\ &\leq MP(A_n \cap \{X \leq M\}) + \int_{\{X>M\}} XdP \\ &\leq MP(A_n) + \int_{\{X>M\}} XdP.\end{aligned}$$

Now since $P(A_n) \to 0$, we have

$$\limsup_n \int_{A_n} XdP \leq \int_{\{X>M\}} XdP.$$

Taking $M \to \infty$ and using part (a) gives

$$\limsup_n \int_{A_n} XdP \leq 0 \leq \liminf_n \int_{A_n} XdP$$

where the second inequality follows from X nonnegative. Thus, $\lim_{n\to\infty} \int_{A_n} XdP = 0$. Lastly, if X is not nonnegative, then

$$\limsup_n \int_{A_n} XdP = \lim_{n\to\infty} \int_{A_n} X^+dP - \lim_{n\to\infty} \int_{A_n} X^-dP = 0 - 0 = 0$$

and similarly for $\liminf_n$, giving the result.

(c) Show $\int_A |X| dP = 0$ iff $P(A \cap \{|X| > 0\}) = 0$.

Solution: Assume $\int_A |X| dP = 0$. Recall that any nonnegative random variable, say $|X| I_A$, is the limit of a monotone increasing sequence of simple random variables. Then,

$$|X| I_A \geq \sum_{k=1}^{n2^n} a_k I_{A \cap A_k} + n I_{A \cap \{|X| > n\}}$$

where we define $a_k = \frac{k-1}{2^n}$ and $A_k = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$. Taking expectations gives

$$0 \stackrel{\star}{=} E[|X| I_A] \geq \sum_{k=1}^{n2^n} a_k P(A \cap A_k) + n P(A \cap \{|X| > n\}).$$

where we have $\stackrel{\star}{=}$ by assuming $\int_A |X| dP = 0$. Therefore, $P(A \cap A_k) = 0$ for all $k = 1, ..., n2^n$ and $P(A \cap \{|X| > n\}) = 0$. By summing over $k \geq 2$ and adding $P(A \cap \{|X| > n\}) = 0$, we have $P(A \cap \{|X| \geq 2^{-n}\}) = 0$. Since $A \cap \{|X| \geq 2^{-n}\}$ are monotone increasing sets in n, we conclude

$$P(A \cap \{|X| > 0\}) = \lim_{n \to \infty} P(A \cap \{|X| \geq 2^{-n}\}) = \lim_{n \to \infty} 0 = 0.$$

Conversely, assume that $P(A \cap \{|X| > 0\}) = 0$. Define $A_n = A \cap \{|X| > 1/n\}$. Note that these sets are monotone increasing and

$$A \cap \{|X| > 0\} = \bigcup_{n=1}^{\infty} A_n.$$

Therefore, we have that

$$P\{A \cap \{|X| > 0\}\} = \lim_{n \to \infty} P(A_n) = 0.$$

Now, by part (b) and the monotone convergence theorem,

$$\int_A |X| dP = \int_\Omega |X| I(A \cap \{|X| > 0\}) dP = \int_\Omega \lim_{n \to \infty} |X| I(A_n) dP$$
$$\stackrel{MCT}{=} \lim_{n \to \infty} \int_\Omega |X| I(A_n) dP = \lim_{n \to \infty} \int_{A_n} |X| dP \stackrel{(b)}{=} 0.$$

This concludes the result.

(d) If $X \in L_2$, show $\text{Var}(X) = 0$ implies $P(X = E(X)) = 1$.

Solution: First note that

$$\int_\Omega (X - E[X])^2 dP = \text{Var}(X) = 0.$$

Then, by part (c), $P((X - E[X])^2 > 0) = 0$, i.e. $P(X = E[X]) = 1$ as desired.

(e) Suppose that $(\Omega, \mathcal{B}, P)$ is a probability space and $A_i \in \mathcal{B}, i = 1, 2$. Define the distance $d \colon \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ by $d(A_1, A_2) = P(A_1 \Delta A_2)$. Prove that if $A_n, A \in \mathcal{B}$ and $d(A_n, A) \to 0$, then $\int_{A_n} X dP \to \int_A X dP$ so that the map $A \to \int_A X dP$ is continuous.

Solution: First note that

$$P(A_n \Delta A) = P(A_n \cap A^c) + P(A \cap A_n^c).$$

Then since $P(A_n \Delta A) \to 0$, $P(A_n \cap A^c) \to 0$ and $P(A \cap A_n^c) \to 0$. Thus, by part (b),

$$\begin{aligned}
\int_{A_n} X dP - \int_A X dP &= \int_{A_n \cap A} X dP + \int_{A_n \cap A^c} X dP - \int_{A \cap A_n} X dP - \int_{A \cap A_n^c} X dP \\
&= \int_{A_n \cap A^c} X dP - \int_{A \cap A_n^c} X dP \\
&\to 0 - 0 = 0.
\end{aligned}$$

Likewise for $\int_A X dP - \int_{A_n} X dP$, and hence we have $|\int_{A_n} X dP - \int_A X dP| \to 0$. This proves the result.

Problem 5.9

Use Fubini's theorem to show for a distribution function $F(x)$

$$\int_{\mathbb{R}} \big(F(x+a) - F(x)\big) dx = a,$$

where dx can be interpreted as Lebesgue measure.

Solution: Note that $x < t < x + a$ implies $t - a < x < t$. Therefore, by Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}} \big(F(x+a) - F(x)\big) dx &= \int_{\mathbb{R}} \int_x^{x+a} f(t) dt dx = \int_{\mathbb{R}} \int_{t-a}^{t} f(t) dx dt \\
&= \int_{\mathbb{R}} f(t) \int_{t-a}^{t} dx dt = \int_{\mathbb{R}} a f(t) dt = a.
\end{aligned}$$

Problem 5.20

For a random variable X with distribution F, define the moment generating function $\phi(\lambda)$ by $\phi(\lambda) = E(e^{\lambda X})$. Let $\Lambda = \{\lambda \in \mathbb{R} \colon \phi(\lambda) < \infty\}$ and set $\lambda_\infty = \sup \Lambda$. Lastly, define the measure F_λ by $F_\lambda(I) = \int_I \frac{e^{\lambda x}}{\phi(\lambda)} F(dx), \lambda \in \Lambda$.

(a) Prove that $\phi(\lambda) = \int_{\mathbb{R}} e^{\lambda x} F(dx)$.

Solution: This follows directly from the fact that $E[g(X)] = \int_{\mathbb{R}} g(x) F(dx)$.

(b) Prove for λ in the interior of Λ that $\phi(\lambda) > 0$ and $\phi(\lambda)$ is continuous on the interior of Λ.

Solution: Suppose $\phi(\lambda) = 0$. This implies that $E[e^{\lambda X}] = 0$, i.e. $e^{\lambda x} = 0$ almost everywhere. However, this function is only 0 at $\pm\infty$ and so this cannot happen. Therefore, $\phi(\lambda) > 0$. Next we show that $\phi(\lambda)$ is continuous on the interior of Λ. Let λ be in the interior of Λ. Then, there exists an $\epsilon > 0$ such that $(\lambda - \epsilon, \lambda + \epsilon) \in \Lambda$. Now, assume $\lambda_n \to \lambda$. Then, for δ where $0 < \delta < \epsilon$, there exists an n_0 such that $|\lambda - \lambda_n| < \delta$ for all $n \geq n_0$. That is, $\lambda - \delta < \lambda_n < \lambda + \delta$ for all $n \geq n_0$. Using this, we have

$$e^{\lambda_n x} \leq e^{(\lambda+\delta)x} + e^{(\lambda-\delta)x}$$

for all $n \geq n_0$. Since $0 < \delta < \epsilon$, $\lambda + \delta$ and $\lambda - \delta$ are in Λ. Therefore, we can use the dominated convergence to obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{\lambda_n x} F(dx) = \int_{\mathbb{R}} \lim_{n\to\infty} e^{\lambda_n x} F(dx) = \int_{\mathbb{R}} e^{\lambda x} F(dx).$$

(c) Give an example where (i) $\lambda_\infty \in \Lambda$ and (ii) $\lambda_\infty \notin \Lambda$.

Solution: Consider the density function

$$f(x) = \frac{ce^{-x}}{(1+x)^2} I(x > 0),$$

where c is the normalizing constant to ensure this is a valid probability density function. Then, certainly $\lambda_\infty = 1$. But,

$$\int_0^\infty \frac{c}{(1+x)^2} dx = c$$

and so $\lambda_\infty \in \Lambda$. Now consider

$$f(x) = \frac{ce^{-x}}{(1+x)} I(x > 0),$$

where c is a different normalizing constant than before. Again, $\lambda_\infty = 1$ but this time

$$\int_0^\infty \frac{1}{(1+x)} dx = \infty$$

and so $\lambda_\infty \notin \Lambda$.

(d) If F has a density f, verify F_λ has a density f_λ. What is f_λ? (Note that the family $\{f_\lambda, \lambda \in \Lambda\}$ is an exponential family of densities.)

Solution: By the definition of F_λ given above,

$$F_\lambda(I) = \int_I \frac{e^{\lambda x}}{\phi(\lambda)} F(dx) = \int_I \frac{e^{\lambda x}}{\phi(\lambda)} f(x) dx.$$

This implies that $f_\lambda(x) = \frac{e^{\lambda x}}{\phi(\lambda)} f(x)$.

(e) If $F(I) = 0$, show $F_\lambda(I) = 0$ as well for I a finite interval and $\lambda \in \Lambda$.

Solution: Given $F(I) = 0$, we have

$$F(I) = \int_I f(x)dx = 0$$

which implies that $f = 0$ almost everywhere. Consequently, $f_\lambda = 0$ almost everywhere and so $F_\lambda(I) = 0$.

Problem 6.12

Let $\{X_n\}$ be a sequence of random variables.

(a) If $X_n \xrightarrow{P} 0$, then for any $p > 0$,

$$\frac{|X_n|^p}{1+|X_n|^p} \xrightarrow{P} 0 \quad (6.21) \quad \text{and} \quad E\left(\frac{|X_n|^p}{1+|X_n|^p}\right) \to 0 \quad (6.22).$$

Solution: For $0 < \epsilon < 1$,

$$P\left\{\frac{|X_n|^p}{1+|X_n|^p} \geq \epsilon\right\} = P\left\{|X_n| \geq \sqrt[p]{\frac{\epsilon}{1-\epsilon}}\right\} \to 0$$

since $X_n \xrightarrow{P} 0$. This shows (6.21). To show (6.22), define the set $A_n = \{|X_n| < \epsilon\}$. Then,

$$\begin{aligned} E\left(\frac{|X_n|^p}{1+|X_n|^p}\right) &= \int_{A_n} \frac{|X_n|^p}{1+|X_n|^p} dP + \int_{A_n^c} \frac{|X_n|^p}{1+|X_n|^p} dP \\ &\leq \int_{A_n} |X_n|^p dP + \int_{A_n^c} dP \\ &< \epsilon P(A_n) + P(A_n^c). \end{aligned}$$

Now, since ϵ was arbitrary, we have

$$E\left(\frac{|X_n|^p}{1+|X_n|^p}\right) \leq P(A_n^c) \to 0$$

as $n \to \infty$ since $X_n \xrightarrow{P} 0$. This shows (6.22).

(b) If (6.21) holds for some $p > 0$, then $X_n \xrightarrow{P} 0$.

Solution: For $\epsilon > 0$, we have

$$P\{|X_n| \geq \epsilon\} = P\left\{\frac{|X_n|^p}{1+|X_n|^p} \geq \frac{\epsilon^p}{1+\epsilon^p}\right\} \to 0$$

as $n \to \infty$ because of (6.21).

(c) Suppose $p > 0$. Show $X_n \xrightarrow{P} 0$ iff (6.22).

Solution: The forward direction has been proven in part (a). We prove the reverse direction by way of contraposition. Assume there exists an $\epsilon_0 > 0$ such that $\lim_{n\to\infty} P(A_n) > 0$, where $A_n = \{|X_n| \geq \epsilon_0\}$. Then,

$$\begin{aligned}\lim_{n\to\infty} E\left(\frac{|X_n|^p}{1+|X_n|^p}\right) &= \lim_{n\to\infty}\int_\Omega \frac{|X_n|^p}{1+|X_n|^p}dP \geq \lim_{n\to\infty}\int_{A_n} \frac{|X_n|^p}{1+|X_n|^p}dP \\ &\geq \lim_{n\to\infty}\int_{A_n}\frac{\epsilon_0^p}{1+\epsilon_0^p}dP = \lim_{n\to\infty}\frac{\epsilon_0^p}{1+\epsilon_0^p}P(A_n) > 0.\end{aligned}$$

This proves the result.

Problem 6.23

A classical transform result says the following: Suppose $u_n \geq 0$ and $u_n \to u$ as $n \to \infty$. For $0 < s < 1$, define the generating function

$$U(s) = \sum_{n=0}^{\infty} u_n s^n.$$

Show that $\lim_{s\to 1}(1-s)U(s) = u$ by the following relatively painless method which uses convergence in probability: Let $T(s)$ be a geometric random variable satisfying $P(T(s) = n) = (1-s)s^n$. Then $T(s) \xrightarrow{P} \infty$. What is $E(u_{T(s)})$?

Solution: First see that

$$(1-s)U(s) = (1-s)\sum_{n=0}^{\infty} u_n s^n = \sum_{n=0}^{\infty} u_n(1-s)s^n = E(u_{T(s)}).$$

Therefore, it suffices to show $E(u_{T(s)}) \to u$ as $s \to 1$. Consider

$$\begin{aligned}|E(u_{T(s)}) - u| &= \left|\sum_{n=0}^{\infty} u_n P(T(s)=n) - u\right| = \left|\sum_{n=0}^{\infty}(u_n - u)P(T(s)=n)\right| \\ &\leq \sum_{n=0}^{\infty}|u_n - u|P(T(s)=n).\end{aligned}$$

Since $u_n \to u$, then for any $\epsilon > 0$, there exists an n_0 such that $|u_n - u| < \epsilon$ for all $n \geq n_0$. Take $M = \max\{|u_i - u|\colon i = 1, ..., n_0 - 1\}$. Then, we have

$$|E(u_{T(s)}) - u| < M\sum_{k=0}^{n_0-1} P(T(s)=k) + \epsilon\sum_{k=n_0}^{\infty} P(T(s)=k) \leq M\sum_{k=0}^{n_0-1}(1-s)s^k + \epsilon.$$

Now taking $s \to 1$, we have $\lim_{s\to 1}|E(u_{T(s)}) - u| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude $\lim_{s\to 1}|E(u_{T(s)}) - u| = 0$, which is what we needed to show.

Problem 6.30

For a random variable X, define $||X||_\infty = \sup\{x\colon P(|X| > x) > 0\}$. Let L_∞ be the set of all random variables X for which $||X||_\infty < \infty$.

(a) Show that for a random variable X and $1 < p < q < \infty$,

$$0 \le ||X||_1 \le ||X||_p \le ||X||_q \le ||X||_\infty.$$

Solution: First note that $||X||_1 \le ||X||_p$ follows immediately from Holder's inequality. Let $Z = |X|^p$. Since $q > p$, then $q/p > 1$ and so by Holder's inequality,

$$E[|X|^p] = E[|Z \cdot 1|] \le E[|Z|^{q/p}]^{p/q} \cdot 1 = E[|X|^q]^{p/q}.$$

Raising both sides to the $1/p$ gives $||X||_p \le ||X||_q$. For the last inequality, define the set $A = \{|X| > ||X||_\infty\}$. Then, $P(A) = 0$, because otherwise $||X||_\infty$ wouldn't be the supremum. Therefore, we have

$$E[|X|^q] = \int_\Omega |X|^q dP = \int_{A^c} |X|^q dP \le \int_{A^c} ||X||_\infty^q dP = ||X||_\infty^q.$$

Raising both sides to $1/q$ gives $||X||_q \le ||X||_\infty$ and the problem is complete.

(b) For $1 < p < q < \infty$, show $L_\infty \subset L_q \subset L_p \subset L_1$.

Solution: This follows immediately from part (a).

(c) Show Holder's inequality holds in the form $E(|XY|) \le ||X||_1 ||Y||_\infty$.

Solution: Consider the set $A = \{|Y| > ||Y||_\infty\}$ from part (a). Then,

$$\begin{aligned}
E(|XY|) &= \int_\Omega |XY| dP = \int_{A^c} |X||Y| dP \\
&\le \int_{A^c} |X|||Y||_\infty dP = ||Y||_\infty \int_{A^c} |X| dP \\
&= ||Y||_\infty \int_\Omega |X| dP = ||X||_1,
\end{aligned}$$

which proves the result.

(d) Show Minkowski's inequality holds in the form $||X + Y||_\infty \le ||X||_\infty + ||Y||_\infty$.

Solution: Define the sets

$$\begin{aligned}
A &= \{x\colon P(|X| > x) > 0\} \\
B &= \{x\colon P(|Y| > x) > 0\} \\
AB &= \{x\colon P(|X + Y| > x) > 0\}.
\end{aligned}$$

Let $a \in AB$. Then, $P(|X + Y| > a) > 0$, and so $P(|X| + |Y| > a) > 0$. By denseness,

$$\{|X| + |Y| > a\} = \bigcup_{\substack{r_x, r_y \in \mathbb{Q} \\ r_x + r_y > a}} \{|X| > r_x, |Y| > r_y\}.$$

Since the left hand side has positive probability, there must exist an r_x, r_y such that $P(|X| > r_x, |Y| > r_y) > 0$ and $a < r_x + r_y$. This implies $P(|X| > r_x) > 0$ and $P(|Y| > r_y) > 0$, i.e. $r_x \in A$ and $r_y \in B$. Lastly, taking supremums of $a < r_x + r_y$ over each of the three sets will give $||X + Y||_\infty \leq ||X||_\infty + ||Y||_\infty$.